Change Detection under Compound Gaussian Assumptions

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Definition of problem

Notations:

In the scope of this paper, the following notations will be used: Lower-case (resp. Upper-case) bold letters denotes vectors (resp. matrices). $\mathbb{R}^p$ and $\mathbb{C}^p$ is the sets of real and complex p-dimensional vectors. $\mathbb{B} = \{0, 1\}$. Θ is an arbitrary parameter space. $\mathbf{0}_p$ is the p-dimensional null vector. For any given matrice, $\bullet^{\mathrm{T}}$, $\bullet^{\mathrm{H}}$ represent the transpose and transpose conjugate operators. $\mathrm{Tr}(\bullet)$, $|\bullet|$ and $\| \bullet \|_{l_2}$ are the trace, determinant and l_2 norm operators. $\bullet^{-1}$ is the inverse operation. Given a scalar valued function f, $\frac{\partial f}{\partial \bullet}$ denotes the gradient of w.r.t $\bullet$ arranged in a column. $\mathbf{x}$ will always represent a random vector of size p. Any subscript or superscript serves to indicate a peculiar observation. The notation $\bullet^{(\epsilon)}$ is used to denotes a given date. $\mathbf{\Sigma}$ will always be an Hermitian matrix of size p^2. The symbol $\sim$ means "distributed as". H_0 and H_1 denote both possible hypothesis in a binary hypothesis test scheme.

$\mathcal{CCG}$ Model

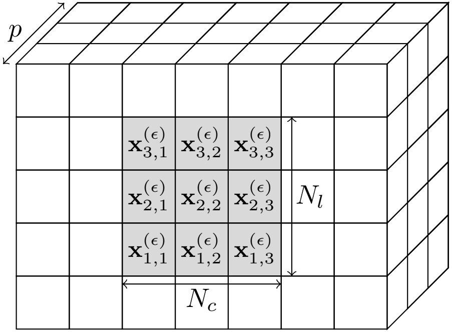

Figure: Illustration of local data selection ($N_l = N_c = p = 3$) for detection test. The gray area corresponds to $\mathfrak{W}_{\mathbf{x}}^{(\epsilon)}$ and the central pixel ($\mathbf{x}_{2,2}^{(\epsilon)}$) is the test pixel.

We define $\mathbb{W} = \{1,\dots,N_l\} \times \{1,\dots,N_c\}$ and denote the observations on the window as $\mathbf{x}_{l,c}^{(\epsilon)}$. The subscripts $(l,c) \in \mathbb{W}$ serve to identify the pixel and $\epsilon \in \mathbb{B}$ the date of observation. $N = N_l \times N_c$ and $\forall \epsilon \in \mathbb{B}$, $\mathfrak{W}_{\mathbf{x}}^{(\epsilon)} = \left\{\mathbf{x}_{l,c}^{(\epsilon)}\right\}_{(l,c)\in\mathbb{W}}$.
The CCG model for each pixel is summarized trough the formulation:

$$\mathbf{x}_{l,c}^{(\epsilon)} \sim \mathcal{CCG}(\mathbf{0}_p, \tau_{l,c}^{(\epsilon)}, \mathbf{\Sigma}^{(\epsilon)})$$

$$p_{\mathbf{x}}^{\mathcal{CCG}}(\mathbf{x};\tau,\mathbf{\Sigma}) = \frac{1}{\pi^p\ |\mathbf{\Sigma}|\ \tau^p} \exp\left(-\frac{\mathbf{x}^{\mathrm{H}}\,\mathbf{\Sigma}^{-1}\,\mathbf{x}}{\tau}\right). \tag{1}$$

We define $\mathfrak{T}_{\mathbf{x}}^{(\epsilon)} = \left\{\tau_{l,c}^{(\epsilon)}\right\}_{(l,c)\in\mathbb{W}}$.

Derivation of Detectors

Given independent identically distributed observations $\{\mathbf{x}_k\}_{k\in\{1,\dots,N\}}$, a pdf model $p_{\mathbf{x}}(\mathbf{x};\theta)$ parametrised by a known parameter $\theta \in \Theta$, the LRT is given by:

$$\hat{\Lambda}_{\mathrm{LRT}} = \frac{\mathcal{L}(\mathbf{x}_1,\dots,\mathbf{x}_N;\theta/\mathrm{H}_1)}{\mathcal{L}(\mathbf{x}_1,\dots,\mathbf{x}_N;\theta/\mathrm{H}_0)} \underset{\mathrm{H}_0}{\overset{\mathrm{H}_1}{\gtrless}} \lambda \tag{2}$$

where $\mathcal{L}$ is the likelihood of the observations given by

$$\mathcal{L}(\mathbf{x}_1,\dots,\mathbf{x}_N;\theta) = \prod_{k=1}^{N} p_{\mathbf{x}_k}(\mathbf{x}_k;\theta).$$

When the parameter θ is unknown, it can be replaced in eq. (2) by an estimate. This method is called Two-step LRT. The Maximum Likelihood Estimator (MLE) is usually used:

$$\hat{\theta}_{\mathrm{MLE}} = \underset{\theta}{\operatorname{argmax}}\, \mathcal{L}(\mathbf{x}_1,\dots,\mathbf{x}_N;\theta). \tag{3}$$

MLE of $\mathcal{CCG}$ dsitribution

The MLE for texture and covariance matrix for CCG distributions are:

$$\begin{aligned}
&\forall (l,c) \in \mathbb{W},\ \forall \epsilon \in \mathbb{B},\\
&\qquad \widehat{\tau_{l,c}^{(\epsilon)}} = \frac{1}{p} {\mathbf{x}_{l,c}^{(\epsilon)}}^{\mathrm{H}} {\boldsymbol{\Sigma}^{(\epsilon)}}^{-1} \mathbf{x}_{l,c}^{(\epsilon)},\\
&\qquad \widehat{\boldsymbol{\Sigma}_{\mathrm{MLE}}^{(\epsilon)}} = \frac{p}{N} \sum_{(l,c)\in\mathbb{W}} \frac{\mathbf{x}_{l,c}^{(\epsilon)} {\mathbf{x}_{l,c}^{(\epsilon)}}^{\mathrm{H}}}{{\mathbf{x}_{l,c}^{(\epsilon)}}^{\mathrm{H}} {\widehat{\boldsymbol{\Sigma}_{\mathrm{MLE}}^{(\epsilon)}}}^{-1} \mathbf{x}_{l,c}^{(\epsilon)}}.
\end{aligned} \tag{4}$$

Proof: For any $(l,c) \in \mathbb{W}$, $\epsilon \in \mathbb{B}$, we solve $\partial\mathcal{L}/\partial\theta = 0$ using $\theta = \left\{\tau_{l,c}^{(\epsilon)}, \boldsymbol{\Sigma}^{(\epsilon)}\right\}$. The cumbersome calculus yields the result presented in eq. (4).

The GLRT differs from the LRT in the sense that it suppose *a priori* that the parameter of the distribution are unknown: Given independent identically distributed observations $\{\mathbf{x}_k\}_{k\in\{1,\dots,N\}}$, a pdf model $p_{\mathbf{x}}(\mathbf{x};\theta)$ parametrised by an unknown parameter $\theta \in \Theta$, the GLRT is given by:

$$\hat{\Lambda}_{\text{GLRT}} = \frac{\max\limits_{\theta} \mathcal{L}(\mathbf{x}_1, \dots, \mathbf{x}_N; \theta/\mathrm{H}_1)}{\max\limits_{\theta} \mathcal{L}(\mathbf{x}_1, \dots, \mathbf{x}_N; \theta/\mathrm{H}_0)} \underset{\mathrm{H}_0}{\overset{\mathrm{H}_1}{\gtrless}} \lambda \tag{5}$$

where $\mathcal{L}$ is the likelihood of the observations. Here, in contrast to the Two-Step LRT, the optimization is done on the likelihood given the conditions of hypothesis $\mathrm{H}_{\epsilon\in\mathbb{B}}$. The maximization can be done using MLE estimates under the given hypothesis.

In this detection problem, we want to detect a change corresponding conjointly to a change in power and in the covariance matrix. This differs from the classic Gaussian detection test (where the power is implicitly tested trough the covariance matrix) as the heterogeneity of the texture on the window of observations is taken into account in the model.
The detection problem, denoted $\mathrm{Pb}_{\mathrm{MT}}$, is as follows:

$$\forall \epsilon \in \mathbb{B}, \forall (l,c) \in \mathbb{W}, \qquad \begin{cases} \mathrm{H}_0 : & \mathbf{x}_{l,c}^{(\epsilon)} \sim \mathcal{CCG}(\mathbf{0}_p, \tau_{l,c}^{(0)}, \mathbf{\Sigma}^{(0)}) \\ \mathrm{H}_1 : & \mathbf{x}_{l,c}^{(\epsilon)} \sim \mathcal{CCG}(\mathbf{0}_p, \tau_{l,c}^{(\epsilon)}, \mathbf{\Sigma}^{(\epsilon)}) \end{cases} \tag{6}$$

Under H_0 hypothesis, the textures are the same for both dates $\epsilon \in \mathbb{B}$ and are denoted $\tau_{l,c}^{(0)}$.

LRT

The LRT ratio under hypothesis of problem Pb_{MT} is the following:

$$\hat{\Lambda}_{\text{LRT}}^{\text{MT}} = \frac{\left|\mathbf{\Sigma}^{(0)}\right|^N}{\left|\mathbf{\Sigma}^{(1)}\right|^N} \prod_{(l,c)\in\mathbb{W}} \frac{{\tau_{l,c}^{(0)}}^p}{{\tau_{l,c}^{(1)}}^p} \times \exp\left(\text{Tr}\Big(\sum_{(l,c)\in\mathbb{W}} \Big(\frac{{\mathbf{\Sigma}^{(0)}}^{-1}}{\tau_{l,c}^{(0)}} - \frac{{\mathbf{\Sigma}^{(1)}}^{-1}}{\tau_{l,c}^{(1)}}\Big)\mathbf{x}_{l,c}^{(1)}{\mathbf{x}_{l,c}^{(1)}}^{\text{H}}\Big)\right) \underset{\text{H}_0}{\overset{\text{H}_1}{\gtrless}} \lambda \tag{7}$$

By replacing the texture and covariance parameters in eq. (7) by their MLE estimates presented in eq. (4), we obtain the following detector:

$$\hat{\Lambda}_{2-\text{LRT}}^{\text{MT}} = \frac{\left|\widehat{\mathbf{\Sigma}_{\text{MLE}}^{(0)}}\right|^N}{\left|\widehat{\mathbf{\Sigma}_{\text{MLE}}^{(1)}}\right|^N} \prod_{(l,c)\in\mathbb{W}} \frac{\left({\mathbf{x}_{l,c}^{(0)}}^{\text{H}}\, {\widehat{\mathbf{\Sigma}_{\text{MLE}}^{(0)}}}^{-1} \mathbf{x}_{l,c}^{(0)}\right)^p}{\left({\mathbf{x}_{l,c}^{(1)}}^{\text{H}}\, {\widehat{\mathbf{\Sigma}_{\text{MLE}}^{(1)}}}^{-1} \mathbf{x}_{l,c}^{(1)}\right)^p} \underset{\text{H}_0}{\overset{\text{H}_1}{\gtrless}} \lambda \tag{8}$$

The GLRT ratio under hypothesis of problem $\mathrm{Pb}_{\mathrm{MT}}$ is the following:

$$\hat{\Lambda}_{\mathrm{GLRT}}^{\mathrm{MT}} = \frac{\left|\widehat{\boldsymbol{\Sigma}_{\mathrm{MT}}^{(0)}}\right|^{2N}}{\left|\widehat{\boldsymbol{\Sigma}_{\mathrm{MLE}}^{(0)}}\right|^{N}\left|\widehat{\boldsymbol{\Sigma}_{\mathrm{MLE}}^{(1)}}\right|^{N}} \times \prod_{(l,c)\in\mathbb{W}} \frac{\left(\mathbf{x}_{l,c}^{(0)\,\mathrm{H}}\widehat{\boldsymbol{\Sigma}_{\mathrm{MT}}^{(0)}}^{-1}\mathbf{x}_{l,c}^{(0)} + \mathbf{x}_{l,c}^{(1)\,\mathrm{H}}\widehat{\boldsymbol{\Sigma}_{\mathrm{MT}}^{(0)}}^{-1}\mathbf{x}_{l,c}^{(1)}\right)^{2p}}{2^{2p}\left(\mathbf{x}_{l,c}^{(0)\,\mathrm{H}}\widehat{\boldsymbol{\Sigma}_{\mathrm{MLE}}^{(0)}}^{-1}\mathbf{x}_{l,c}^{(0)}\right)^{p}\left(\mathbf{x}_{l,c}^{(1)\,\mathrm{H}}\widehat{\boldsymbol{\Sigma}_{\mathrm{MLE}}^{(1)}}^{-1}\mathbf{x}_{l,c}^{(1)}\right)^{p}} \underset{\mathrm{H}_0}{\overset{\mathrm{H}_1}{\gtrless}} \lambda \quad (9)$$

with:

$$\widehat{\boldsymbol{\Sigma}_{\mathrm{MT}}^{(0)}} = \frac{p}{N}\sum_{(l,c)\in\mathbb{W}} \frac{\mathbf{x}_{l,c}^{(0)}\mathbf{x}_{l,c}^{(0)\,\mathrm{H}} + \mathbf{x}_{l,c}^{(1)}\mathbf{x}_{l,c}^{(1)\,\mathrm{H}}}{\mathbf{x}_{l,c}^{(0)\,\mathrm{H}}\widehat{\boldsymbol{\Sigma}_{\mathrm{MT}}^{(0)}}^{-1}\mathbf{x}_{l,c}^{(0)} + \mathbf{x}_{l,c}^{(1)\,\mathrm{H}}\widehat{\boldsymbol{\Sigma}_{\mathrm{MT}}^{(0)}}^{-1}\mathbf{x}_{l,c}^{(1)}}. \quad (10)$$

Proof of GLRT (1/2)

We treat both hypothesis $\mathrm{H}_{\epsilon\in\mathbb{B}}$ separately:

- Under H_0: The log-likelihood function is:

$$\log\mathcal{L} = \log\mathcal{L}^{(0)} + \log\mathcal{L}^{(1)} \text{ with: } \mathcal{L}^{(\epsilon\in\mathbb{B})} = p_{\mathfrak{W}_{\mathbf{x}}^{(\epsilon)}}(\mathfrak{W}_{\mathbf{x}}^{(\epsilon)}; \mathfrak{T}_{\mathbf{x}}^{(0)}, \mathbf{\Sigma}^{(0)}/\mathrm{H}_0)$$
$$= -2N\log\left(\left|\mathbf{\Sigma}^{(0)}\right|\right) - \sum_{(l,c)\in\mathbb{W}} 2p\log(\tau_{l,c}^{(0)}) - \sum_{k}\sum_{\epsilon\in\mathbb{B}} \frac{\mathbf{x}_{l,c}^{(\epsilon)}\mathbf{\Sigma}^{(0)}\mathbf{x}_{l,c}^{(\epsilon)\,\mathrm{H}}}{\tau_{l,c}^{(0)}} \quad (11)$$

Solving $\forall(l,c)\in\mathbb{W}$, $\frac{\partial}{\partial\tau_{l,c}^{(0)}}\log\mathcal{L} = 0$ yields:

$$\forall(l,c)\in\mathbb{W},\ \widehat{\tau_{l,c}^{(0)}} = \frac{1}{2p}\sum_{\epsilon\in\mathbb{B}} \mathbf{x}_{l,c}^{(\epsilon)}\mathbf{\Sigma}^{(0)}\mathbf{x}_{l,c}^{(\epsilon)\,\mathrm{H}} \quad (12)$$

Solving $\frac{\partial}{\partial\mathbf{\Sigma}^{(0)}}\log\mathcal{L} = 0$ yields the expression at eq. (10).
$\left(\frac{\partial\log(|\mathbf{M}|)}{\partial\mathbf{M}} = \mathbf{M}^{-1},\ \frac{\partial\mathbf{x}^H\mathbf{M}^{-1}\mathbf{x}}{\partial\mathbf{M}} = -\mathbf{x}\,\mathbf{x}^H\,\mathbf{M}^{-2}\right)$

Proof of GLRT (2/2)

- Under H_1: The log-likelihood function is:

$$\log\mathcal{L} = \log\mathcal{L}^{(0)} + \log\mathcal{L}^{(1)} \text{ with: } \mathcal{L}^{(\epsilon\in\mathbb{B})} = p_{\mathfrak{W}_{\mathbf{x}}^{(\epsilon)}}(\mathfrak{W}_{\mathbf{x}}^{(\epsilon)}; \mathfrak{T}_{\mathbf{x}}^{(\epsilon)}, \boldsymbol{\Sigma}^{(\epsilon)}/\mathrm{H}_0)$$

$$= -2N\log(\left|\boldsymbol{\Sigma}^{(\epsilon)}\right|) - \sum_{(l,c)\in\mathbb{W}} 2p\log(\tau_{l,c}^{(\epsilon)}) - \sum_k \sum_{\epsilon\in\mathbb{B}} \frac{\mathbf{x}_{l,c}^{(\epsilon)}\boldsymbol{\Sigma}^{(\epsilon)}{\mathbf{x}_{l,c}^{(\epsilon)}}^{\mathrm{H}}}{\tau_{l,c}^{(\epsilon)}} \quad (13)$$

Solving $\forall(l,c)\in\mathbb{W}$, $\frac{\partial}{\partial\tau_{l,c}^{(\epsilon)}}\log\mathcal{L} = 0$ yields:

$$\forall(l,c)\in\mathbb{W},\ \widehat{\tau_{l,c}^{(\epsilon)}} = \frac{1}{p}\mathbf{x}_{l,c}^{(\epsilon)}\boldsymbol{\Sigma}^{(\epsilon)}{\mathbf{x}_{l,c}^{(\epsilon)}}^{\mathrm{H}} \quad (14)$$

Solving $\frac{\partial}{\partial\boldsymbol{\Sigma}^{(\epsilon)}}\log\mathcal{L} = 0$ yields $\widehat{\boldsymbol{\Sigma}_{\mathrm{MLE}}^{(\epsilon)}}$.

Then by replacing the unknown parameters by their estimate we have maximised the ratio, resulting in the expression given at eq. (9).

In this next detection problem, we want to detect changes in the local correlations between the pixels arbitrarily to their relative power. This scheme is intended for applications in which an alteration in the power is not a significant change (for example two images of a scene with different calibrations). In those situations, the problem $\mathrm{Pb_{MT}}$ is not suited. The detection problem, denoted $\mathrm{Pb_{MG}}$, is as follows:

$$\forall \epsilon \in \mathbb{B}, \forall (l,c) \in \mathbb{W},\quad \begin{cases} \mathrm{H}_0: & \mathbf{x}_{l,c}^{(\epsilon)} \sim \mathcal{CCG}(\mathbf{0}_p, \tau_{l,c}^{(\epsilon)}, \mathbf{\Sigma}^{(0)}) \\ \mathrm{H}_1: & \mathbf{x}_{l,c}^{(\epsilon)} \sim \mathcal{CCG}(\mathbf{0}_p, \tau_{l,c}^{(\epsilon)}, \mathbf{\Sigma}^{(\epsilon)}) \end{cases} \tag{15}$$

Under both hypothesis, no constraint is given on the texture parameters.

LRT

The LRT ratio under hypothesis of problem $\mathrm{Pb}_{\mathrm{MG}}$ is the following:

$$\hat{\Lambda}_{\mathrm{LRT}}^{\mathrm{MG}} = \frac{\left|\boldsymbol{\Sigma}^{(0)}\right|^N}{\left|\boldsymbol{\Sigma}^{(1)}\right|^N} \exp\left(\mathrm{Tr}((\boldsymbol{\Sigma}^{(0)}{}^{-1} - \boldsymbol{\Sigma}^{(1)}{}^{-1}) p \sum_{(l,c)\in\mathbb{W}} \frac{\mathbf{x}_{l,c}^{(1)} \mathbf{x}_{l,c}^{(1)}{}^{\mathrm{H}}}{\tau_{l,c}^{(1)}})\right) \underset{\mathrm{H}_0}{\overset{\mathrm{H}_1}{\gtrless}} \lambda \tag{16}$$

By replacing the texture and covariance parameters in eq. (16) by their MLE estimates presented in eq. (4), we obtain the following detector:

$$\hat{\Lambda}_{2-\mathrm{LRT}}^{\mathrm{MG}} = \frac{\left|\widehat{\boldsymbol{\Sigma}_{\mathrm{MLE}}^{(0)}}\right|^N}{\left|\widehat{\boldsymbol{\Sigma}_{\mathrm{MLE}}^{(1)}}\right|^N} \exp\left(p \sum_{(l,c)\in\mathbb{W}} \frac{\mathbf{x}_{l,c}^{(1)}{}^{\mathrm{H}} \widehat{\boldsymbol{\Sigma}_{\mathrm{MLE}}^{(0)}}^{-1} \mathbf{x}_{l,c}^{(1)}}{\mathbf{x}_{l,c}^{(1)}{}^{\mathrm{H}} \widehat{\boldsymbol{\Sigma}_{\mathrm{MLE}}^{(1)}}^{-1} \mathbf{x}_{l,c}^{(1)}} - pN\right) \underset{\mathrm{H}_0}{\overset{\mathrm{H}_1}{\gtrless}} \lambda \tag{17}$$

The GLRT ratio under hypothesis of problem $\mathrm{Pb}_{\mathrm{MG}}$ is the following:

$$\hat{\Lambda}_{\mathrm{GLRT}}^{\mathrm{MG}} = \frac{\left|\widehat{\mathbf{\Sigma}_{\mathrm{MG}}^{(0)}}\right|^{2N}}{\left|\widehat{\mathbf{\Sigma}_{\mathrm{MLE}}^{(0)}}\right|^{N}\left|\widehat{\mathbf{\Sigma}_{\mathrm{MLE}}^{(1)}}\right|^{N}} \times \prod_{(l,c)\in\mathbb{W}} \frac{\left(\mathbf{x}_{l,c}^{(0)\,\mathrm{H}}\widehat{\mathbf{\Sigma}_{\mathrm{MG}}^{(0)}}^{-1}\mathbf{x}_{l,c}^{(0)}\right)^{p}\left(\mathbf{x}_{l,c}^{(1)\,\mathrm{H}}\widehat{\mathbf{\Sigma}_{\mathrm{MG}}^{(0)}}^{-1}\mathbf{x}_{l,c}^{(1)}\right)^{p}}{\left(\mathbf{x}_{l,c}^{(0)\,\mathrm{H}}\widehat{\mathbf{\Sigma}_{\mathrm{MLE}}^{(0)}}^{-1}\mathbf{x}_{l,c}^{(0)}\right)^{p}\left(\mathbf{x}_{l,c}^{(1)\,\mathrm{H}}\widehat{\mathbf{\Sigma}_{\mathrm{MLE}}^{(1)}}^{-1}\mathbf{x}_{l,c}^{(1)}\right)^{p}} \underset{\mathrm{H}_0}{\overset{\mathrm{H}_1}{\gtrless}} \lambda \tag{18}$$

with:

$$\widehat{\mathbf{\Sigma}_{\mathrm{MG}}^{(0)}} = \frac{p}{2N}\sum_{(l,c)\in\mathbb{W}}\sum_{\epsilon\in\mathbb{B}} \frac{\mathbf{x}_{l,c}^{(\epsilon)}\mathbf{x}_{l,c}^{(\epsilon)\,\mathrm{H}}}{\mathbf{x}_{l,c}^{(\epsilon)\,\mathrm{H}}\widehat{\mathbf{\Sigma}_{\mathrm{MG}}^{(0)}}^{-1}\mathbf{x}_{l,c}^{(\epsilon)}}. \tag{19}$$

Finally, we propose a last detection scheme which is similar to Pb_{MG} as the detection is done solely on the covariance matrix but with an equality constraint on the textures between the two dates. It corresponds to stable zones with an alteration in the scatterers' correlations.
The detection problem, denoted Pb_{MC} , is as follows:

$$\forall \epsilon \in \mathbb{B}, \forall (l,c) \in \mathbb{W}, \\ \begin{cases} \text{H}_0 : & \mathbf{x}_{l,c}^{(\epsilon)} \sim \mathcal{CCG}(\mathbf{0}_p, \tau_{l,c}^{(0)}, \mathbf{\Sigma}^{(0)}) \\ \text{H}_1 : & \mathbf{x}_{l,c}^{(\epsilon)} \sim \mathcal{CCG}(\mathbf{0}_p, \tau_{l,c}^{(0)}, \mathbf{\Sigma}^{(\epsilon)}) \end{cases} \tag{20}$$

Under both hypothesis, the texture parameters are supposed equal.

LRT

The LRT ratio under hypothesis of problem $\mathrm{Pb}_{\mathrm{MC}}$ is the following:

$$\hat{\Lambda}_{\mathrm{LRT}}^{\mathrm{MC}} = \frac{\left|\boldsymbol{\Sigma}^{(0)}\right|^N}{\left|\boldsymbol{\Sigma}^{(1)}\right|^N} \exp\left(\mathrm{Tr}\Big(\big({\boldsymbol{\Sigma}^{(0)}}^{-1} - {\boldsymbol{\Sigma}^{(1)}}^{-1}\big)p \sum_{(l,c)\in\mathbb{W}} \frac{\mathbf{x}_{l,c}^{(1)}{\mathbf{x}_{l,c}^{(1)}}^{\mathrm{H}}}{\tau_{l,c}^{(0)}}\Big)\right) \underset{\mathrm{H}_0}{\overset{\mathrm{H}_1}{\gtrless}} \lambda \tag{21}$$

By replacing the texture and covariance parameters in eq. (21) by their MLE estimates presented in eq. (4), we obtain the following detector:

$$\hat{\Lambda}_{2-\mathrm{LRT}}^{\mathrm{MC}} = \frac{\left|\widehat{\boldsymbol{\Sigma}_{\mathrm{MLE}}^{(0)}}\right|^N}{\left|\widehat{\boldsymbol{\Sigma}_{\mathrm{MLE}}^{(1)}}\right|^N} \exp\left(p \sum_{(l,c)\in\mathbb{W}} \frac{{\mathbf{x}_{l,c}^{(1)}}^{\mathrm{H}} {\widehat{\boldsymbol{\Sigma}_{\mathrm{MLE}}^{(0)}}}^{-1} \mathbf{x}_{l,c}^{(1)}}{{\mathbf{x}_{l,c}^{(0)}}^{\mathrm{H}} {\widehat{\boldsymbol{\Sigma}_{\mathrm{MLE}}^{(0)}}}^{-1} \mathbf{x}_{l,c}^{(0)}} - pN\right) \underset{\mathrm{H}_0}{\overset{\mathrm{H}_1}{\gtrless}} \lambda \tag{22}$$

The GLRT ratio under hypothesis of problem $\mathrm{Pb}_{\mathrm{MC}}$ is the following:

$$\hat{\Lambda}_{\mathrm{GLRT}}^{\mathrm{MC}} = \frac{\left|\widehat{\boldsymbol{\Sigma}_{\mathrm{MC}}^{(0,\mathrm{H}_0)}}\right|^{2N}}{\left|\widehat{\boldsymbol{\Sigma}_{\mathrm{MC}}^{(0,\mathrm{H}_1)}}\right|^{N}\left|\widehat{\boldsymbol{\Sigma}_{\mathrm{MC}}^{(1,\mathrm{H}_1)}}\right|^{N}} \times \prod_{(l,c)\in\mathbb{W}} \frac{\left(\mathbf{x}_{l,c}^{(0)}{}^{\mathrm{H}}\widehat{\boldsymbol{\Sigma}_{\mathrm{MC}}^{(0,\mathrm{H}_0)}}^{-1}\mathbf{x}_{l,c}^{(0)} + \mathbf{x}_{l,c}^{(1)}{}^{\mathrm{H}}\widehat{\boldsymbol{\Sigma}_{\mathrm{MC}}^{(0,\mathrm{H}_0)}}^{-1}\mathbf{x}_{l,c}^{(1)}\right)^{p}}{\left(\mathbf{x}_{l,c}^{(0)}{}^{\mathrm{H}}\widehat{\boldsymbol{\Sigma}_{\mathrm{MC}}^{(0,\mathrm{H}_1)}}^{-1}\mathbf{x}_{l,c}^{(0)} + \mathbf{x}_{l,c}^{(1)}{}^{\mathrm{H}}\widehat{\boldsymbol{\Sigma}_{\mathrm{MC}}^{(1,\mathrm{H}_1)}}^{-1}\mathbf{x}_{l,c}^{(1)}\right)^{p}} \underset{\mathrm{H}_0}{\overset{\mathrm{H}_1}{\gtrless}} \lambda \tag{23}$$

with:

$$\widehat{\boldsymbol{\Sigma}_{\mathrm{MC}}^{(0,\mathrm{H}_0)}} = \frac{p}{N}\sum_{(l,c)\in\mathbb{W}} \frac{\mathbf{x}_{l,c}^{(0)}\mathbf{x}_{l,c}^{(0)}{}^{\mathrm{H}} + \mathbf{x}_{l,c}^{(1)}\mathbf{x}_{l,c}^{(1)}{}^{\mathrm{H}}}{\mathbf{x}_{l,c}^{(0)}{}^{\mathrm{H}}\widehat{\boldsymbol{\Sigma}_{\mathrm{MC}}^{(0,\mathrm{H}_0)}}^{-1}\mathbf{x}_{l,c}^{(0)} + \mathbf{x}_{l,c}^{(1)}{}^{\mathrm{H}}\widehat{\boldsymbol{\Sigma}_{\mathrm{MC}}^{(0,\mathrm{H}_0)}}^{-1}\mathbf{x}_{l,c}^{(1)}}, \tag{24}$$

$$\widehat{\boldsymbol{\Sigma}_{\mathrm{MC}}^{(\epsilon,\mathrm{H}_1)}} = \frac{p}{N}\sum_{(l,c)\in\mathbb{W}} \frac{\mathbf{x}_{l,c}^{(\epsilon)}\mathbf{x}_{l,c}^{(\epsilon)}{}^{\mathrm{H}}}{\mathbf{x}_{l,c}^{(0)}{}^{\mathrm{H}}\widehat{\boldsymbol{\Sigma}_{\mathrm{MC}}^{(0,\mathrm{H}_1)}}^{-1}\mathbf{x}_{l,c}^{(0)} + \mathbf{x}_{l,c}^{(1)}{}^{\mathrm{H}}\widehat{\boldsymbol{\Sigma}_{\mathrm{MC}}^{(1,\mathrm{H}_1)}}^{-1}\mathbf{x}_{l,c}^{(1)}}. \tag{25}$$

Properties

Geodesic Convexity

Let $\mathcal{M}$ be an arbitrary manifold. For each pair $q_0, q_1 \in \mathcal{M}$, we define a geodesic $q_t^{q_0,q_1}$ for $t \in [0,1]$. For simplicity, we omit the superscripts and assume q_0 and q_1 are understood from the context.

Definition: A real valued function f with domain $\mathcal{M}$ is g-convex if $f(q_t) \leq tf(q_1) + (1-t)f(q_0)$ for any $q_0, q_1 \in \mathcal{M}$.

Proposition: Any local minima of a g-convex function over $\mathcal{M}$ is a global minima.

Consider the manifold $\mathbb{S}_{++}^p$ of positive definite matrices. With each $\mathbf{\Sigma}_0, \mathbf{\Sigma}_1 \in \mathbb{S}_{++}^p$, we associate the following geodesic:

$$\mathbf{\Sigma}_t = \mathbf{\Sigma}_0^{\frac{1}{2}} \left(\mathbf{\Sigma}_0^{-\frac{1}{2}} \mathbf{\Sigma}_1 \mathbf{\Sigma}_0^{-\frac{1}{2}} \right)^t \mathbf{\Sigma}_0^{\frac{1}{2}}, \, t \in [0,1] \tag{26}$$

Some Lemmas on $\mathbb{S}_{++}^p$

Lemma 1: Let $\mathbf{h} \in \mathbb{R}^p$ and $a \in \pm 1$. The function

$$f(\mathbf{\Sigma}) = \mathbf{h}^{\mathrm{T}} \mathbf{\Sigma}^a \mathbf{h} \tag{27}$$

is g-convex in $\mathbf{\Sigma} \in \mathbb{S}_{++}^p$.

Lemma 2: Let $a \in \pm 1$ and $\mathbf{H}_i \in \mathbb{R}^{q,p}$ for $i = 1, \ldots, n$ be a set of matrices whose qp columns span $\mathbb{R}^q$. The function

$$f(\mathbf{\Sigma}) = \log \left| \sum_{i=1}^{n} \mathbf{H}_i \mathbf{\Sigma}^a {\mathbf{H}_i}^{\mathrm{T}} \right| \tag{28}$$

is g-convex in $\mathbf{\Sigma} \in \mathbb{S}_{++}^p$.

- Whiten the data and covariance in the detectors' expression and show that it dosen't depend on the data. And take into account normalsiaiton constraint for fixed point estimators.
- Find a group of invariance that contain the normalisation constraint on fixed point estimators ?